
A General-Purpose Approach to Computer-Aided Dynamic Analysis of a Flexible Helicopter

Om P. Agrawal

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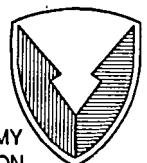
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Om P. Agrawal, Southern Illinois University at Carbondale, Carbondale, Illinois

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N88-26709 #

SUMMARY

This report describes a general-purpose mathematical model for dynamic analysis of a helicopter consisting of flexible and/or rigid bodies (components) that may undergo large translations and large rotations. In this formulation, two sets of generalized coordinates are used, namely rigid-body and elastic. The rigid body coordinates define the location and the orientation of a body coordinate frame (global frame) with respect to an inertial frame. The elastic coordinates are introduced using a finite element approach in order to model flexible components with complex shapes. The compatibility conditions between two adjacent elements of a flexible body are imposed using a Boolean matrix, whereas the compatibility conditions between two adjacent bodies are imposed using the Lagrange multiplier approach. The virtual work approach is used to derive the equations of motion. This leads to a system of differential and algebraic equations that are highly nonlinear due to large rotations of the components of the system. Some numerical schemes to solve the resulting set of equations are also discussed.

The formulation presented is general. When compared with a hard-wired formulation, this approach has several advantages. 1) The flexible model of each component in the system is processed identically. Therefore, only one set of subroutines is required to generate several matrices such as inertia and stiffness matrices. 2) The constant submatrices associated with a component remain constant throughout the simulation and the sparsity of these component matrices is preserved because the matrices associated with a component do not depend on the generalized coordinates of the other components. 3) The form of constraint equations depends upon the type of kinematic joint and involves the generalized coordinates of the two components participating in the kinematic joint. Therefore, a library of constraint elements can be developed and the kinematic constraint between two adjacent bodies can be imposed in an automated fashion. 4) This formulation is capable of generating equations of motion for a system with any number of flexible and/or rigid bodies connected in an arbitrary fashion. 5) The Lagrange multipliers which are a byproduct of this formulation provide the reaction forces and torques at the joints.

1. INTRODUCTION

Analysis of a helicopter consisting of interconnected rigid and/or flexible components, each of which may undergo large translations and large rotations, requires a mathematical model that accounts for nonlinear kinematic constraints, nonlinear couplings between rigid motion and elastic deformations, and inertia variations resulting from elastic deformations. In mathematical models developed in the past (refs. 1-6), the nonlinear kinematic constraints were removed explicitly and the equations of motion were developed in terms of a minimum number of generalized coordinates. Such models are, although quite appealing because they lead to a set of minimum number of differential equations, quite involved in respect of their mathematical formulations. In contrast to automated coupling, these models are often limited in their flexibility to represent every possible kinematic joint in terms of the minimum number of generalized coordinates considered, and are also limited in their expandability to a new system since they are based on a specific topology of the system. Furthermore, these models destroy the sparsity and the constant matrices appearing in the component equations of motion, which makes it difficult to take advantage of the sparsity and modal analysis for space economy and dimension reduction. Finally, these models often introduce singularities in the models which require special care.

It is clear that a new mathematical model for analysis of a helicopter is desired. This report develops a general purpose model for analysis of a helicopter and identifies features of this model which makes it superior to the previously developed models.

2. GENERALIZED COORDINATES AND VIRTUAL WORK DUE TO INERTIA FORCES

In order to specify the configuration of a body (component) of a helicopter system, it is necessary to define a set of generalized coordinates such that the absolute position of a point in the body can be defined in terms of those generalized coordinates. As shown in figure 1, XYZ is the inertial frame and $X^i Y^i Z^i$ is the coordinate frame (or the global frame) associated with body i. The body i may be a fuselage, a rotor, or any other component of the helicopter system. In the discussion to follow, the coordinate system associated with body i will be called frame i. Using figure 1, the position vector of point p, r_p , can be written as (refs. 7, 8),

$$r_p = R^i + A^i s \quad (2.1)$$

where $R^i (= [x^i, y^i, z^i])$ is the position vector of the origin of frame i with respect to the inertial frame, A^i is the orthonormal transformation matrix from frame i to the inertial frame, and s is the position vector of point p with respect to frame i. Vector s may be written as

$$s = s_0 + u \quad (2.2)$$

where s_0 is the undeformed location of point p with respect to frame i, and u is the vector field of elastic deformations. Body i is idealized as a finite collection of structural elements having n_i number of nodes

and n_d number of nodal degrees of freedom. Assuming elastic deformations to be small, the vector field u can be approximated as

$$u = N^i q^i \quad (2.3)$$

where q^i is a $n_d \times 1$ vector of nodal elastic deformations of body i and N^i is the shape function. Note that this model allows one to have different numbers of degrees of freedom at each node. Although the virtual work expression due to inertia forces (and subsequently the equations of motion) can be developed in terms of field variable u , the finite element approach is used here to allow modeling of flexible components with complex shapes.

Substituting equations (2.2) and (2.3) into equation (2.1), the position vector of point p can be approximated as

$$r_p = R^i + A^i(s_o + u) \quad (2.4a)$$

$$= R^i + A^i(s_o + N^i q^i) \quad (2.4b)$$

Differentiating equation (2.4) with respect to time, the velocity vector of point p is obtained.

$$\dot{r}_p = \dot{R}^i + \dot{A}^i s + A^i \dot{u} \quad (2.5a)$$

$$= \dot{R}^i + \dot{A}^i s + A^i N^i \dot{q}^i \quad (2.5b)$$

where (\cdot) denotes differentiation with respect to time. The second term in the right-hand side of equation (2.5) can be written as

$$\dot{A}^i s = \sum_{j=1}^l \frac{\partial A^i}{\partial \theta_j} \cdot \dot{\theta}_j s = B^i(\theta^i, s) \dot{\theta}^i \quad (2.6)$$

where $\theta_j (j = 1, \dots, l)$ are the rotational degrees of freedom of the i th global frame, and θ^i is the vector of $\theta_j (j = 1, \dots, l)$, i.e.,

$$\theta^i = \begin{bmatrix} \theta_1 \\ \vdots \\ \theta_l \end{bmatrix} \quad (2.7)$$

These rotational degrees of freedom can be Euler angles, Euler-like angles, Euler parameters, or other rotational coordinates including quasi-rotational coordinates. In most cases $l = 3$. However, in the case

of Euler parameters, $l = 4$. Selection of rotational degrees of freedom has been an issue of much debate. For small rotations, most rotational degrees of freedom do not present any problem. For large rotations, however, certain rotational degrees of freedom may lead to singularity problems.

Since the matrix B^i is linear in its second argument, it follows that

$$B^i(\theta, u + v) = B^i(\theta, u) + B^i(\theta, v) \quad (2.8)$$

This property of matrix B^i is used later to isolate the rigid body, the elastic body, and the coupling terms. The matrix B^i is linear in its first argument if Euler parameters are used to define the rotational degrees of freedom. Using equation (2.6), the velocity vector of point p can be expressed as

$$\dot{r}_p = \dot{R}^i + B^i(\theta, s)\dot{\theta}^i + A^i\dot{u} \quad (2.9a)$$

$$= \dot{R}^i + B^i(\theta, s)\dot{\theta}^i + A^iN^i\dot{q}^i \quad (2.9b)$$

Define

$$P^i = \begin{bmatrix} R^i \\ \theta^i \\ q^i \end{bmatrix} \quad (2.10)$$

to be the vector of generalized coordinates of body i. The first two components of P^i represent rigid body degrees of freedom and q^i represents the elastic coordinates. Using equation (2.10), the velocity of point p becomes

$$\dot{r}_p = \begin{bmatrix} I & B^i(\theta, s) & A^iN^i \end{bmatrix} \dot{P}^i \quad (2.11)$$

where I is a 3x3 identity matrix. Differentiating equation (2.11) with respect to time, one obtains the acceleration vector of point p as

$$\ddot{r}_p = \begin{bmatrix} I & B^i(\theta, s) & A^iN^i \end{bmatrix} \ddot{P}^i + \dot{B}^i(\theta, s)\dot{\theta}^i + \dot{A}^iN^i\dot{q}^i \quad (2.12a)$$

$$= \begin{bmatrix} I & B^i(\theta, s) & A^iN^i \end{bmatrix} \ddot{P}^i + \dot{B}^i(\theta, s)\dot{\theta}^i + B^i(\theta, \dot{u})\dot{\theta}^i \quad (2.12b)$$

Similarly the virtual displacement of point p can be given by

$$\delta r_p = \delta R^i + \delta A^i s + A^i \delta u \quad (2.13a)$$

$$= \delta R^i + B^i(\theta, s) \delta \theta^i + A^i N^i \delta q^i \quad (2.13b)$$

where $\delta(\cdot)$ represents the variation of a parameter.

The virtual work due to inertia forces can now be written as

$$\delta W_I^i = \int_{m^i} \delta r_p^T \ddot{r}_p dm^i \quad (2.14b)$$

where dm^i is the differential mass of a small volume at point p, m^i is the total mass of body i, and the superscript T represents the transpose of a vector (or a matrix). Note that the virtual work due to inertia forces contains three distinct virtual work expressions, namely, virtual work due to: (1) virtual translations of frame i, (2) virtual rotations of frame i, and (3) virtual elastic deformations. Substituting equations (2.12) and (2.13) into equation (2.14) and using equation (2.10), the three virtual work expressions due to inertia forces can be written as

1) Virtual work due to virtual translation of frame i:

$$\begin{aligned} \delta W_{I(tran)}^i &= \delta R^{iT} \int_{m^i} \begin{bmatrix} I & B^i(\theta, s) & A^i N^i \end{bmatrix} \ddot{P}^i dm^i \\ &+ \delta R^{iT} \int_{m^i} (\dot{B}^i(\theta, s) \dot{\theta}^i + B^i(\theta, \dot{u}) \dot{\theta}^i) dm^i \end{aligned} \quad (2.15a)$$

2) Virtual work due to virtual rotation of frame i:

$$\begin{aligned} \delta W_{I(rot)}^i &= \delta \theta^{iT} \int_{m^i} \begin{bmatrix} B^{iT}(\theta, s) & B^{iT}(\theta, s) B^i(\theta, s) & B^{iT}(\theta, s) A^i N^i \end{bmatrix} \ddot{P}^i dm^i \\ &+ \delta \theta^{iT} \int_{m^i} (B^{iT}(\theta, s) \dot{B}^i(\theta, s) \dot{\theta}^i + B^{iT}(\theta, s) B^i(\theta, \dot{u}) \dot{\theta}^i) dm^i \end{aligned} \quad (2.15b)$$

3) Virtual work due to virtual elastic deformation:

$$\begin{aligned} \delta W_{I(elas)}^i &= \delta q^{iT} \int_{m^i} \begin{bmatrix} N^{iT} A^{iT} & N^{iT} A^{iT} B^i(\theta, s) & N^{iT} N^i \end{bmatrix} \ddot{P}^i dm^i \\ &+ \delta q^{iT} \int_{m^i} (N^{iT} A^{iT} \dot{B}^i(\theta, s) \dot{\theta}^i + N^{iT} A^{iT} B^i(\theta, \dot{u}) \dot{\theta}^i) dm^i \end{aligned} \quad (2.15c)$$

Combining equations (2.15a) to (2.15c), the expression for virtual work due to inertia forces of body i becomes

$$\delta W_I^i = \delta P^{iT} M^i \ddot{P}^i + \delta P^{iT} F^i \quad (2.16)$$

where M^i is the mass matrix of body i and F^i is a vector of nonlinear terms that includes generalized forces due to centrifugal and Coriolis accelerations. Matrix M^i and vector F^i are, respectively, given by

$$M^i = \begin{bmatrix} M_{rr}^i & M_{r\theta}^i & M_{rq}^i \\ M_{r\theta}^{iT} & M_{\theta\theta}^i & M_{\theta q}^i \\ M_{rq}^{iT} & M_{\theta q}^{iT} & M_{qq}^i \end{bmatrix} \quad (2.17)$$

and

$$F^i = \int_{m^i} \begin{bmatrix} \dot{B}^i(\theta, s) + B^i(\theta, \dot{u}) \\ B^{iT}(\theta, s) \dot{B}^i(\theta, s) + B^{iT}(\theta, s) B^i(\theta, \dot{u}) \\ N^{iT} A^{iT} \dot{B}^i(\theta, s) + N^{iT} A^{iT} B^i(\theta, \dot{u}) \end{bmatrix} \dot{\theta}^i dm^i \quad (2.18)$$

where

$$M_{rr}^i = \int_{m^i} I dm^i = m^i I \quad (2.19)$$

is the diagonal rigid-body translational mass matrix,

$$M_{\theta\theta}^i = \int_{m^i} B^{iT}(\theta, s) B^i(\theta, s) dm^i \quad (2.20)$$

is the rotational inertia matrix,

$$M_{qq}^i = \int_{m^i} N^{iT} N^i dm^i \quad (2.21)$$

is the elastic mass matrix,

$$M_{r\theta}^i = \int_{m^i} B^i(\theta, s) dm^i \quad (2.22)$$

$$M_{rq}^i = \int_{m^i} A^i N^i dm^i = A^i \int_{m^i} N^i dm^i \quad (2.23)$$

and

$$M_{\theta q}^i = \int_{m^i} B^{iT}(\theta, s) A^i N^i dm^i \quad (2.24)$$

are inertia coupling matrices.

The matrices that appear in equation (2.17) (and equations (2.19) to (2.24)) need further examination. Matrices M_{rr}^i and M_{qq}^i are constant matrices and therefore are generated only once. Matrix M_{qq}^i (defined in eq. (2.21)) is the same as the mass matrices which appear in the usual finite element formulation of structural systems. In contrast to the rigid body matrices, matrices $M_{r\theta}^i$ and $M_{\theta\theta}^i$ depend on elastic deformations of the body. Using equations (2.2) and (2.8), matrix $M_{r\theta}^i$ can be written as

$$M_{r\theta}^i = \int_{m^i} B^i(\theta, s_0) dm^i + \int_{m^i} B^i(\theta, u) dm^i \quad (2.25)$$

where the first term on the right hand side is associated with the moment of mass of the undeformed body about its body frame axes and the second term is associated with the moment of mass about the body frame axes due to elastic deformations. The first term is zero if the origin of the undeformed body frame is taken at the center of mass of the body.

Using equations (2.2) and (2.8) once again, the rotational inertia matrix $M_{\theta\theta}^i$ can be written as

$$\begin{aligned} M_{\theta\theta}^i = & \int_{m^i} B^{iT}(\theta, s_0) B^i(\theta, s_0) dm^i + \int_{m^i} (B^{iT}(\theta, s_0) B^i(\theta, u) \\ & + B^{iT}(\theta, u) B^i(\theta, s_0) + B^{iT}(\theta, u) B^i(\theta, u)) dm^i \end{aligned} \quad (2.26)$$

where the first term on the right hand side is the rotational inertia matrix associated with rigid body rotation and the other terms describe contributions due to elastic deformations. Using equation (2.6), it is clear that the shape function and the rotational degrees of freedom in equations (2.22) and (2.23) are isolated from each other and therefore the integral

$$\int_{m^i} N^i dm^i \quad (2.27)$$

is generated only once. Similarly, the rotational degrees of freedom and the elastic deformation in equations (2.18), (2.20), and (2.24) can be isolated. The mathematical manipulations in this case depend on the rotational degrees of freedom selected.

From the above discussion, it is clear that the matrix M^i is highly nonlinear because of large rotations and elastic deformations. Furthermore, rigid body motion and elastic deformations are strongly coupled. An accurate model, which allows the components of the helicopter system to undergo large rotations, must include these nonlinear and inertia coupling effects. Neglecting these effects in helicopter dynamics may result in a significant error in the dynamic response of the helicopter since generalized forces are quite sensitive to generalized coordinates.

The virtual work expression due to inertia forces presented above is general and applies to each body equally. Therefore, the total virtual work due to inertia forces is

$$\delta W_I = \sum_{i=1}^{NB} \delta W_I^i \quad (2.28)$$

where NB is the number of components or bodies in the system. The generalized coordinates of each component need not be independent. If one needs to develop the virtual work δW_I in terms of independent generalized coordinates, then equation (2.28) must be modified accordingly.

The rigid body modes associated with shape function must be eliminated in order to define elastic deformations uniquely. This can be achieved using a specified body frame, a set of mode shapes, or an imposed set of boundary conditions. These three approaches are essentially the same. For example, imposing the condition that the body frame of a beam be rigidly attached to a node at one end of the beam is equivalent to selecting a set of cantilever modes. In this formulation, it is assumed that a body frame is rigidly attached to a node associated with the body. This is not a limitation of this formulation. If another set of body frames is more appropriate or if another set of modes are more desirable, then one can develop a proper transformation relation between nodal elastic variables and minimum elastic coordinates of the body. The transformation matrices appearing in such relations are in general constant and linear, and therefore such transformations can be applied throughout the formulation without any change in the formulation presented in this report.

3. VIRTUAL WORK DUE TO ELASTIC FORCES

The virtual work due to elastic forces in body i is given by (refs. 7, 8)

$$\delta W_e^i = \int_{v^i} \delta \epsilon^{iT} \sigma^i dv^i \quad (3.1)$$

where ϵ^i and σ^i are, respectively, vectors of strain and stress at a point in body i . The virtual work due to elastic forces considered in equation (3.1) is general and includes nonlinear stress-strain relationship and the viscoelastic materials. For simplicity, only a linear stress-strain relation is considered hereafter. Using Hooke's law, the stress-strain relation can be written as

$$\sigma^i = E\epsilon^i \quad (3.2)$$

and for small elastic deformations, the strain displacement relation can be written as

$$\epsilon^i = Du \quad (3.3)$$

where E is the modulus of elasticity matrix and D is a differential operator. Using equation (2.3), the strain displacement relation can be expressed as

$$\epsilon^i = DN^i q^i \quad (3.4)$$

Substituting equations (3.2) and (3.4) into equation (3.1) yields

$$\delta W_e^i = \int_{v^i} \delta q^{iT} (DN^i)^T E (DN^i) q^i dv = \delta q^{iT} K_{ee}^i q^i \quad (3.5)$$

where K_{ee}^i is the elastic stiffness matrix of body i . The stiffness matrix K_{ee}^i is same as the finite-element stiffness matrix appearing in standard linear structural analysis and is a constant matrix. Therefore, one needs to generate this matrix only once. Using the generalized coordinate vector P^i , the virtual work δW_e^i is

$$\delta W_e^i = \delta P^{iT} K^i P^i \quad (3.6)$$

where

$$K^i = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & K_{ee}^i \end{bmatrix} \quad (3.7)$$

There is no stiffness coupling between rigid body and elastic coordinates because the strain depends entirely on elastic deformations.

The total virtual work due to elastic forces is given by

$$\delta W_e = \sum_i \delta W_e^i = \sum_i \delta P^{iT} K^i P^i \quad (3.8)$$

4. VIRTUAL WORK PRODUCED BY EXTERNAL FORCES

The external forces considered here may be point loads, body forces, or surface forces, or a combination of all three. Body forces include the portion of the load attributed to gravity, surface forces may be aerodynamic drag and lift and other surface tractions, and point loads may arise when certain control elements are present. Furthermore, forces due to several other elements, such as translational and/or rotational spring-damper actuators, can also be considered.

Let F_b^i and F_s^i be, respectively, the body force and the surface force acting on body i . The virtual work produced by these forces is given by

$$\delta W_l^i = \int_v F_b^{iT} \cdot \delta r dv^i + \int_s F_s^{iT} \cdot \delta r ds^i \quad (4.1)$$

In the case of point loads, force F_b^i is expressed using a delta function. Substituting the value of the virtual displacement from equation (2.13) into equation (4.1), one finds

$$\begin{aligned} \delta W_l^i &= \left[\int_v F_b^{iT} \begin{bmatrix} I & B^i(\theta, s) & A^i N^i \end{bmatrix} dv^i + \int_s F_s^{iT} \begin{bmatrix} I & B^i(\theta, s) & A^i N^i \end{bmatrix} ds^i \right] \cdot \delta P^i \\ &= \delta P^{iT} \cdot Q^i \end{aligned} \quad (4.2)$$

where

$$Q^i = \int_v \begin{bmatrix} I \\ B^{iT} \\ N^{iT} A^{iT} \end{bmatrix} F_b^i \cdot dv^i + \int_s \begin{bmatrix} I \\ B^{iT} \\ N^{iT} A^{iT} \end{bmatrix} F_s^i \cdot ds^i \quad (4.3)$$

is the vector of generalized forces. In equation (4.3), the first row represents the vector of translational force along the inertial axis, the second row is associated with the torque about the body axis, and the third row is a vector of generalized forces associated with virtual elastic deformations. The torque about the body axes, which is constant in rigid body analysis, is no longer constant due to elastic deformations. The total virtual work due to all external forces acting on the system is given by

$$\delta W_l = \sum_i \delta W_l^i = \sum_i \delta P^{iT} \cdot Q^i \quad (4.4)$$

The virtual work due to the workless constraint forces does not appear in the above expression.

5. FORMULATION METHODS AND KINEMATIC CONSIDERATIONS

Having obtained the virtual work expressions due to inertia forces, internal elastic forces, and external forces, one can proceed to derive the equations of motion of the system. For dynamic equilibrium of the system, the sum of the virtual work due to inertia and elastic forces must be equal to the virtual work due to external forces. This implies

$$\delta W_I + \delta W_e = \delta W_l \quad (5.1)$$

Substituting the values of δW_I , δW_e , and δW_l from equations (2.28), (3.8), and (4.4), leads to

$$\sum_{i=1}^{NB} \delta P^i{}^T [M^i \ddot{P}^i + F^i + K^i P^i - Q^i] = 0 \quad (5.2)$$

If the virtual displacements δP^i ($i = 1, \dots, NB$) are independent of each other, then equation (5.2) is satisfied only if the coefficients of δP^i are zero. This implies

$$M^i \ddot{P}^i + F^i + K^i P^i - Q^i = 0 \quad (i = 1, \dots, NB) \quad (5.3)$$

Equation (5.3), however, is not true if the components are kinematically constrained. For example, in a helicopter, the fuselage and a rotor are kinematically constrained to allow only one relative degree of freedom with respect to each other. Similarly, one may have a universal joint between a rotor blade and the rotor hub. Such kinematic joints impose nonlinear algebraic constraints that must be satisfied. This implies that the generalized coordinates P^i ($i = 1, \dots, NB$) are no longer independent, and the coefficient of δP^i in equation (5.2) can not be set equal to zero. Therefore, equation (5.2) must be modified to account for the kinematic constraints.

The kinematic constraints between adjacent bodies can be incorporated into the equations of motion using one of the following three approaches: 1) the Lagrange multiplier approach, 2) explicit reduction of the equations of motion in terms of independent generalized coordinates, and 3) the hybrid approach.

In the first approach, a set of algebraic constraints that defines the kinematics of the system is developed. This set is then introduced into equation (5.2) using the Lagrange multiplier approach. This leads to a system of differential and algebraic equations that must be solved simultaneously. The structure of the algebraic constraint equations depends on the type of the kinematic constraints (or joints) and is totally independent of the type of system considered. Furthermore, the algebraic equations associated with a kinematic joint are functions of the generalized coordinates of only the two components that are participating in the kinematic joint. Therefore, in this approach, a library of constraints can be developed and the constraints between two adjacent elements can be imposed in an automated fashion.

In the second approach, the differential equations of motion are developed in terms of a set of independent coordinates. In most cases it is difficult to express all virtual work expressions entirely in terms of independent coordinates. Therefore, the dependent virtual displacements and the dependent acceleration in equation (5.2) are replaced, respectively, by independent virtual displacements and independent accelerations, and the mass matrix and other matrices are allowed to be functions of dependent and independent coordinates. This is always possible because virtual displacements are linearly related. Similarly, the components of the acceleration vector are also linearly related. One advantage of this approach is that the dynamics of the system are expressed in terms of the minimum number of second-order differential equations. In this approach, however, the sparsity of the matrices is lost and several submatrices which were constant at the component level do not remain constant in the assembled system of equations. Furthermore, this approach yields special purpose codes of hardwired mathematical models that are generally difficult to update.

In the third approach, the system is divided into a number of groups consisting of one or more components in each group such that no component is a member of two or more groups. The virtual work expressions are then developed for each group in terms of its minimum number of generalized coordinates. Then each group is dynamically coupled to the adjacent group using Lagrange multipliers. Such groups are called superelements. This approach attempts to take advantage of the above two approaches. A set of superelements that accounts for every kinematic joint is not yet available. Furthermore, such code can be used efficiently by the most experienced users only. In this report, only the first approach will be expanded.

6. CONSTRAINT EQUATIONS

In this section, a set of constraint equations for a revolute joint that includes the coupling between a fuselage and a rotor is developed. This is done to illustrate how a particular set of constraints is derived and to show by implication that a library of constraint cases can be derived and treated as constraint elements in the modeling processes.

Figure 2 shows a simplified representation of a fuselage (body i) and a rotor (body j). Let the two points P and Q lie along the axis of the rotor, and be common to the two bodies. The kinematic compatibility condition between the two bodies requires that: (1) the point P on body i and on body j should be connected, and (2) the vector \vec{PQ} in body i and in body j should always be collinear. Mathematically, the first condition implies

$$R^i + A^i(s_{opi} + N^i(P) \cdot q^i) - R^j - A^j(s_{opj} + N^j(P)q^j) = 0 \quad (6.1)$$

where s_{opi} and s_{opj} are, respectively, the undeformed position vector of point P with respect to the global axes of body i and body j , and $N^i(P)$ and $N^j(P)$ are, respectively, the value of the shape function N^i and N^j at point P .

The vector \vec{PQ} in body i is given by

$$(\vec{PQ})^i = A^i((s_{oqi} - s_{opi}) + (N^i(Q) - N^i(P))q^i) \quad (6.2)$$

Similarly, the vector (\vec{PQ}) in body j is

$$(\vec{PQ})^j = A^j((s_{ojj} - s_{opj}) + (N^j(Q) - N^j(P))q^j) \quad (6.3)$$

The second compatibility condition then requires that

$$(\vec{PQ})^i \times (\vec{PQ})^j = 0 \quad (6.4)$$

Equation (6.4) gives three constraint conditions; however, only two of them are independent. Any two conditions from equation (6.4) are sufficient to impose the second compatibility condition. The five constraint equations, (three from equation (6.1) and two from equation (6.4)) define the kinematic constraint between a fuselage and a rotor. This approach allows the elastic deformations at the kinematic joint. If body i and/or body j are rigid then the corresponding constraint equations are obtained by setting the elastic deformation of the rigid body equal to zero. Also points P and Q need not be node points. However, if points P and Q are chosen at two nodes, then equations (6.1) and (6.4) are further simplified.

The constraint equations developed above are general and are applicable to all revolute joints (and therefore it applies to a fuselage and a rotor pair also). Furthermore, these constraints are functions of the generalized coordinates of only the two bodies. Therefore, only one set of constraint equations for each revolute joint is required.

The set of constraint equations developed above is known as a revolute joint constraint element. A helicopter system may include other types of kinematic joints, such as universal joints (e.g., a universal joint between a rotor and a blade), spherical joints, translational joints, cylindrical joints, and spherical-cylindrical joints. Consideration of these joints and several other kinematic joints can be quite important in modeling a helicopter. Constraint elements for these joints can be developed from purely geometrical considerations. For example, a constraint element for a universal joint, as shown in figure 3, can be developed by imposing these conditions (1) point P in body i and in body j should be connected, and (2) the vector \vec{PA} in body i should be normal to the vector \vec{PB} in body j . Constraint equations developed in this manner can be incorporated in a library of nonlinear constraint elements. These elements then can be used to set up automated kinematic constraints between two adjacent bodies.

The complete set of constraint equations will be represented by

$$\phi(P, t) = 0 \quad (6.5)$$

where ϕ is a $m \times 1$ vector of constraints, P is a composite vector of P^i given by

$$P = \begin{bmatrix} P^{1T} & \dots & P^{NB^T} \end{bmatrix}^T \quad (6.6)$$

and t is the time parameter. The time parameter has been included here to emphasize that the constraints may explicitly depend on time.

7. SYSTEM EQUATIONS OF MOTION

In deriving differential equations of motion via the virtual work approach (eq. (5.2)), the requirement for constraints (eq. (6.5)) does not appear until the last step, when the variations of δP^i are considered independent of each other. Recall that the virtual displacements δP^i in equation (5.2) are not independent of each other because of the presence of constraints. Therefore the coefficients of δP^i in equation (5.2) cannot be set to zero. In order to develop the virtual work expression in terms of independent virtual displacements, it was noted that equation (6.5) is equivalent to a differential equation,

$$\sum_i \frac{\partial \phi}{\partial P^i} dP^i + \frac{\partial \phi}{\partial t} dt = 0 \quad (7.1)$$

By definition, the virtual displacement is one in which the time is considered frozen, hence the virtual displacements δP^i must satisfy

$$\sum_i \frac{\partial \phi}{\partial P^i} \delta P^i = 0 \quad (7.2)$$

If equation (7.2) is true, then it is also true that

$$\lambda^T \sum_i \frac{\partial \phi}{\partial P^i} \delta P^i = 0 \quad (7.3)$$

where λ is a $m \times 1$ vector of unknown Lagrange multipliers. Equation (7.3) can be adjoined with equation (5.2), yielding the relation

$$\sum_{i=1}^{NB} \delta P^{iT} \left[M^i \ddot{P}^i + F^i + K^i P^i - Q^i + \left(\frac{\partial \phi}{\partial P^i} \right)^T \lambda \right] = 0 \quad (7.4)$$

Note that δP^i ($i = 1, \dots, NB$) are still not independent. Following an approach presented in reference 9, it can be shown that

$$M^i \ddot{P}^i + F^i + K^i P^i - Q^i + \left(\frac{\partial \phi}{\partial P^i} \right)^T \lambda = 0 \quad (7.5)$$

Equation (7.5) is the desired set of differential equations of motion. If the idealized system has n number of generalized coordinates and m Lagrange multipliers, then equation (7.5) together with equation (6.5) provides $(n + m)$ equations in $(n + m)$ unknowns. Equations (7.5) and (6.5) together define a set of differential and algebraic equations that must be solved simultaneously.

In the present formulation, the matrices M^i , F^i , K^i , and Q^i remain completely unaffected by the kinematic joints appearing in the system. As a result, the constant submatrices appearing in matrices M^i , F^i , K^i , and Q^i remain constant and the sparsity of these matrices is preserved throughout the dynamic simulation. Further, the components of the system can be joined to adjacent elements by selecting proper constraints from the library of constraint elements. This formulation also allows generation of the equations of motion of the system with any number of flexible and/or rigid components. An additional benefit is that the vector

$$\left(\frac{\partial \phi}{\partial P^i} \right)^T \lambda \quad (7.6)$$

provides reaction forces at the kinematic joints.

The derivation of equations of motion presented is general, and one can consider any set of rotational degrees of freedom and arbitrary finite-element shape functions. The specific form of the various matrices appearing in the above formulation will depend on the rotational degrees of freedom and the finite-element shape functions considered.

8. NUMERICAL COMPUTATION METHODS

The mathematical formulation presented in the previous sections leads to a set of differential and algebraic equations. In this section, a brief review of numerical methods used to solve the set of differential and algebraic equations is presented (eqs. 6.5 and 7.5). The numerical schemes considered here are those which have been used extensively in multibody dynamics. These numerical schemes can be divided into the following three groups: schemes which satisfy the constraints explicitly at every step; schemes which attempt to satisfy the constraints implicitly; and hybrid schemes, in which the above two approaches are combined.

Numerical schemes that satisfy the constraints explicitly at every step are as follows: lower upper (LU) factorization technique (LUFT) (ref. 10), singular value decomposition technique (SVD) (refs. 11, 12), QR decomposition technique (QRDT) (ref. 13), null space method (NSM) (refs. 14, 15), and tangent coordinate method (TCM) (ref. 16).

In each of these schemes, the independent coordinates are integrated using a differential-equation solution subroutine, and dependent coordinates are obtained using an iteration scheme such as the Newton-

Raphson scheme. In LUFT, the independent coordinates are identified using LU factorizations of the constraint Jacobian matrix. The theory of differential equations in manifold shows that the solution of the dynamical equations lies on the constraint surface. In SVDT, QRDT, and TCM the tangent coordinates are generated using, respectively, Singular Value Decomposition, QR Decomposition, and Gramm-Schmidt Orthonormalization of the Jacobian matrix.

Schemes that attempt to satisfy the constraints implicitly are: the constraint stabilization technique (CST) (refs. 17, 18), and the penalty function method (PFM) (ref. 19). Some variations of these schemes have also been developed (ref. 20). In these techniques, attempts are made to implicitly satisfy the constraints. This is achieved by introducing artificial damping into the system differential equations of motion.

In the hybrid schemes, the Constraint Stabilization Technique is used to advance the integration from one step to the next step and the constraints are checked at every step. If the constraint violations goes above the error tolerance, then the dependent coordinates are corrected to satisfy the constraints and integration is continued.

It should be mentioned here that the integration schemes considered are those that can be completely isolated from the mathematical formulation of the equations of motion. This is quite important for further extension of the code.

9. CONCLUDING REMARKS

A general mathematical formulation for automated computer-aided analysis of a helicopter system, whose components may undergo large translations and large rotations, has been presented. In this formulation, a helicopter system is considered as a collection of several components (bodies) interconnected by kinematic joints to allow the kinematic degrees of freedom between two adjacent components. A body reference frame is associated with each component. Two sets of generalized coordinates, namely rigid body and elastic, are used to locate a point on the component. The rigid body coordinates define the position and the orientation of the body frame. Each flexible component is idealized as a collection of finite elements rigidly connected at the nodal points. The elastic deformations at the nodal points in a body are defined with respect to the body frame and a transformation matrix from the body frame to the inertial frame is used to define the elastic deformations in the inertial frame. The deformations between two nodes are approximated using shape functions that are defined in the body frame. The generalized coordinates defined above are used to develop expressions for virtual work produced by inertia forces, elastic forces, and external forces. The kinematic constraint conditions between two adjacent bodies are expressed by a set of nonlinear constraint equations. The virtual work approach is used to derive the equations of motion and the Lagrange multiplier approach is used to adjoin the constraint equations to the equations of motion. The model thus combines the traditional finite element approaches that are capable of handling small elastic deformations with the multibody approaches that allow its components to undergo large translations and large rotations.

In the mathematical development, the role of a body axis needs to be emphasized. In this formulation, the elastic deformations between the nodes are approximated using shape functions. This is possible pri-

marily because the strain and the rotation of an element about its axis is small (because of the choice of body frame), and the elastic coordinates are defined with respect to the body frame.

Three features of the transformation matrices that map the body coordinates to inertial coordinates are noteworthy. First, the transformation matrices are used to define the configuration of each component in a single inertial frame. This leads to a single basis-frame in contrast to a vector approach where different vectors may have different basis-frames. Second, the kinematic relations are developed by direct differentiation of the configuration field without consideration of the coordinate systems used to define the parameters. This is significantly different from a vector approach where the total derivative and the derivative with respect to a rotating body axis are not the same. Third, by using transformation matrices, a set of rotational coordinates can be selected that is completely isolated from the other generalized coordinates. The transformation matrices therefore provide flexibility in the choice of rotational coordinates without altering the formulation.

In this model, the generalized coordinates of a component are defined independently of the generalized coordinates of the other components. As a result, a generic component is used to develop a set of mathematical models for inertia effects, internal elastic effects, and effects due to external forces. Each model can be developed in separate modules that can be used for each component in the system. This model of a component has two further advantages. First, the constant matrices associated with a component remain constant, and second, the sparsity of the component matrices is preserved.

In this formulation, two components are connected in an arbitrary fashion. This connection is accomplished by developing a set of nonlinear constraint equations and using a Lagrange multiplier technique. The constraint equations are generated purely from geometrical considerations. A methodology of generating nonlinear constraints for revolute joints and universal joints have been presented. Nonlinear kinematic constraints associated with other types of kinematic couplings, such as articulated joints, spherical joints, translational joints, and other joints can be developed and incorporated as a library of constraint elements. This library then can be used to enforce the kinematic constraints in an automated fashion. The Lagrange multipliers yield the constraint forces at the joints.

Finally, the two most important features of this formulations are the generality of the mathematical model and the modularity of the code. Since this model is general, it allows one to model a helicopter system with arbitrary topological configurations. The generality of this approach allows for theoretical growth of this model without complete reformulation of the problem. This capability is very important when other significant physical aspects of helicopter dynamics must be incorporated in the software as the technology grows and the demand for high accuracy becomes important. The modularity of the code in this approach lends itself to further enhancements. As a result, each module in the code can be developed, checked, and modified independently.

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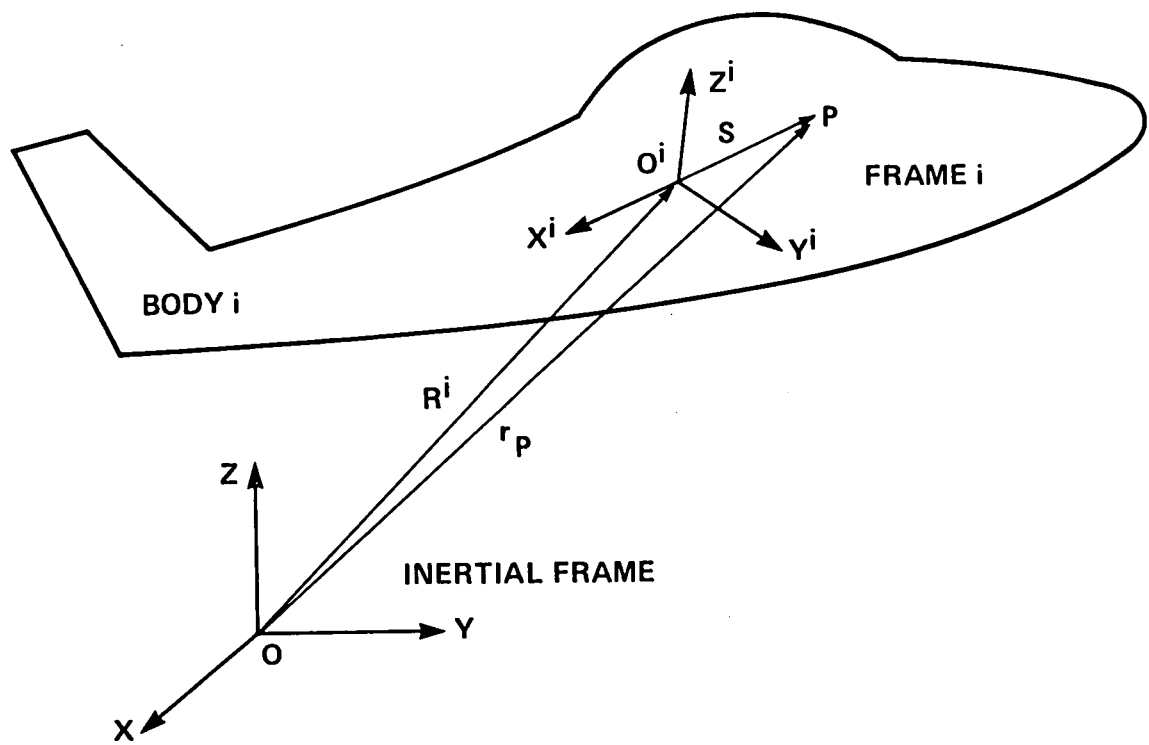


Figure 1.— Reference frames.

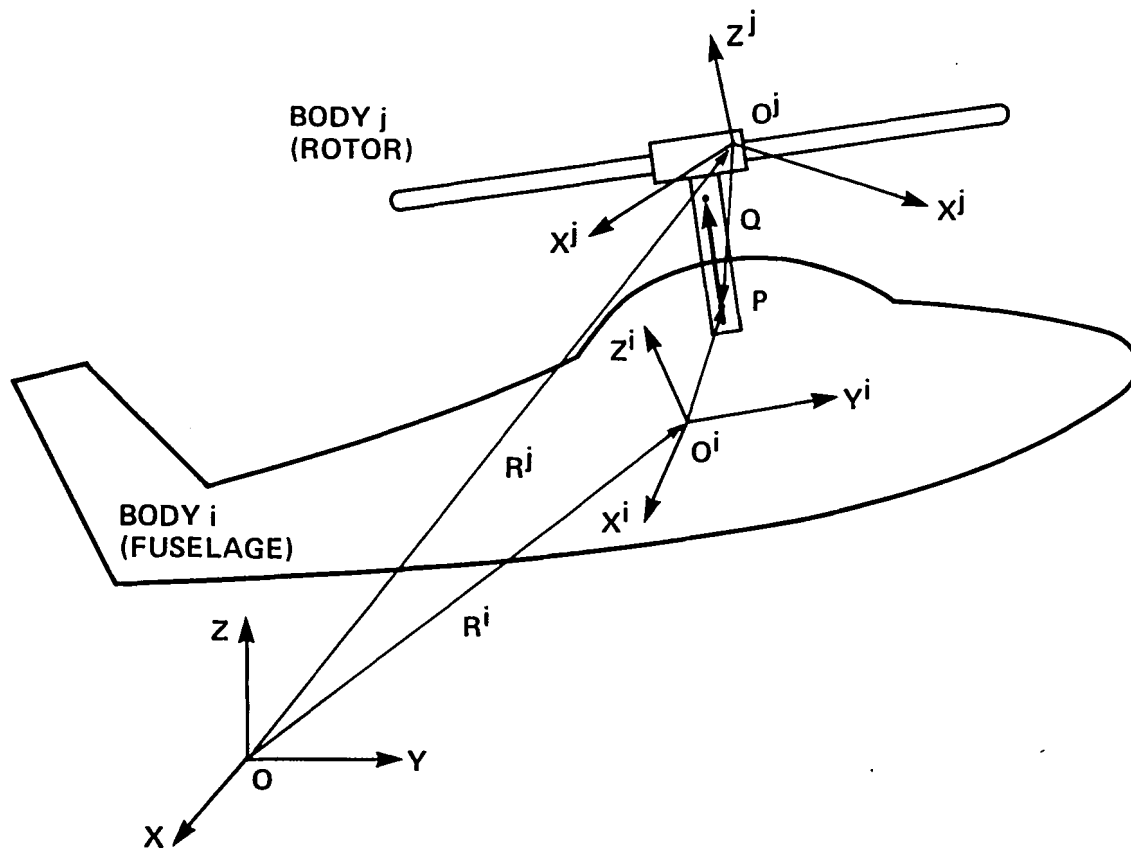


Figure 2.— A kinematic revolute joint between a fuselage and a rotor.

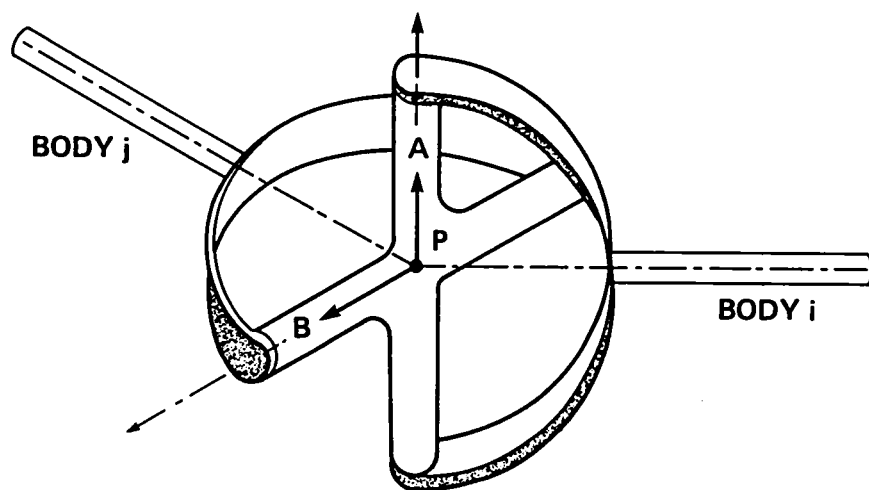


Figure 3.— Universal joint.

Report Documentation Page

1. Report No. NASA TM-101002 USAAVSCOM TR 88-A-006		2. Government Accession No.		3. Recipient's Catalog No.	
4. Title and Subtitle A General-Purpose Approach to Computer Aided Dynamic Analysis of a Flexible Helicopter				5. Report Date July 1988	
				6. Performing Organization Code	
7. Author(s) Om P. Agrawal (Summer Research and Development Program, Southern Illinois University at Carbondale, Carbondale, IL 62901)				8. Performing Organization Report No. A-88183	
				10. Work Unit No. 992-21-01	
9. Performing Organization Name and Address Ames Research Center, Moffett Field, CA 94035 and *Aeroflightdynamics Directorate, U.S. Army Aviation Research and Technology Activity, Ames Research Center, Moffett Field, CA 94035-1099				11. Contract or Grant No.	
				13. Type of Report and Period Covered Technical Memorandum	
12. Sponsoring Agency Name and Address National Aeronautics and Space Administration Washington, DC 20546-0001 and U.S. Army Aviation Systems Command, St. Louis, MO 63120-1798				14. Sponsoring Agency Code	
15. Supplementary Notes Point of Contact: Wendell Stephens, Ames Research Center, MS 215-1 Moffett Field, CA 94035 (415) 694-5794 or FTS 464-5794					
16. Abstract This report describes a general purpose mathematical formulation for dynamic analysis of a helicopter consisting of flexible and/or rigid bodies that undergo large translations and rotations. Rigid body and elastic sets of generalized coordinates are used. The rigid body coordinates define the location and the orientation of a body coordinate frame (global frame) with respect to an inertial frame. The elastic coordinates are introduced using a finite element approach in order to model flexible components. The compatibility conditions between two adjacent elements in a flexible body are imposed using a Boolean matrix. The compatibility conditions between two adjacent bodies are imposed using the Lagrange multiplier approach. Since the form of the constraint equations depends upon the type of kinematic joint and involves only the generalized coordinates of the two participating elements then a library of "constraint elements" can be developed to impose the kinematic constraint in an automated fashion. For the body constraints, the Lagrange multipliers yield the reaction forces and torques of the bodies at the joints. The virtual work approach is used to derive the equations of motion, which are a system of differential and algebraic equations that are highly nonlinear. The formulation presented is general and is compared with hard-wired formulations commonly used in helicopter analysis.					
17. Key Words (Suggested by Author(s)) Dynamic analysis Rotorcraft Lagrange multiplier			18. Distribution Statement Unclassified-Unlimited Subject Category - 39		
19. Security Classif. (of this report) Unclassified	20. Security Classif. (of this page) Unclassified		21. No. of pages 22	22. Price A02	